# Pre-class Warm-up!!!

Let $f(x,y) = 3x^2 - 6xy + 2y^3$. Three questions:

1. What is the Taylor polynomial of degree 2 for $f$ at the point $(x,y) = (0,0)$?

a. $3x^2 - 6xy + 2y^3$.

√ b. $3x^2 - 6xy$

c. $3x^2 - 6xy + 2y^2$

Approach 1: Calculate all $\frac{\partial^2 f}{\partial x \partial y}$ etc, use formula.

Approach 2: b. is the best degree 2 approximation to $f$.

In the book "Taylor polynomial of degree 2" is the same as the "Taylor approximation of order 2"

2. What is the Hessian matrix for $f$ at the point $(x,y) = (0,0)$?

$3 = \frac{1}{2!} \frac{\partial^2 f}{\partial x^2}\Big|_{(0,0)}$ $\quad -6 = \frac{\partial^2 f}{\partial x \partial y}\Big|_{(0,0)}$ $\quad 0 = \frac{1}{2!} \frac{\partial^2 f}{\partial x \partial y}\Big|_{(0,0)}$

a. $\begin{bmatrix} 3 & -6 \\ -6 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 6 & -6 \\ -6 & 12 \end{bmatrix}$

c. $\begin{bmatrix} 3 & -6 \\ -6 & 0 \end{bmatrix}$

√ d. $\begin{bmatrix} 6 & -6 \\ -6 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 6 & -6 \\ -6 & 4 \end{bmatrix}$

- Fact: the Taylor polynomial of degree 2 for $f$ at the point $(x,y) = (1,1)$ is

$$-1 + 3(x-1)^2 - 6(x-1)(y-1) + 6(y-1)^2$$

What is the Hessian matrix for $f$ at $(x,y) = (1,1)$? Solution: matrix b.

Recall the Hessian matrix: For $f(x,y)$ is

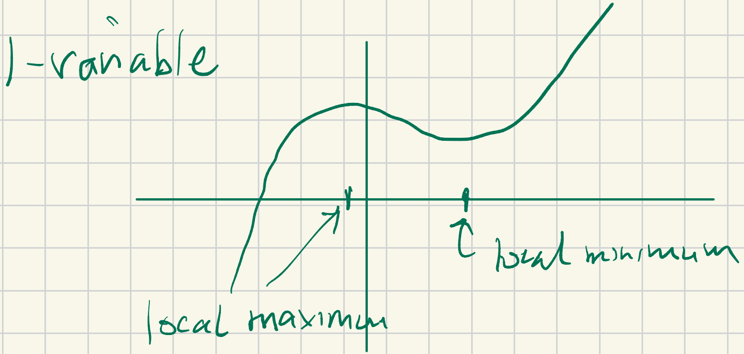
$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

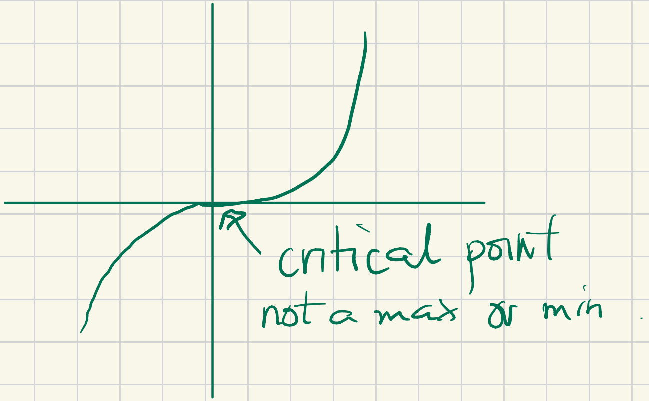
evaluated at $(x_0, y_0)$.

Section 3.3: Extrema of real valued functions
First half: local extrema

We learn

- what do we mean by a local maximum or a local minimum?
- The term: critical point
- How do we find local extrema?
- Just like the 1-variable case, there is a 2nd derivative test, and there are more possibilities than just having a local maximum or minimum: saddle points.



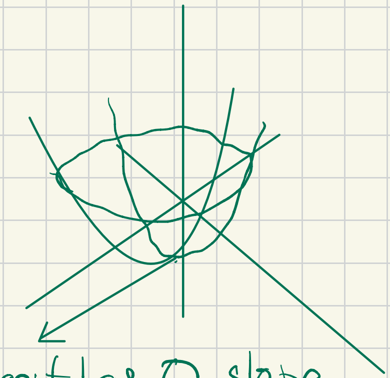
## Definitions:

Let f : R^n -> R, let a be a vector in R^n.

f has a local minimum at a if there is a "neighborhood" around a so that f(a) is the smallest value of f on this neighborhood.

f has a local maximum at a if
Same ... largest ...

f has an ~~extremum~~ local extremum at a means f has a local maximum or minimum at a.




## How do we find local extrema?

Definition: a is a critical point means

$$\left.\frac{\partial f}{\partial x_i}\right|_a = 0 \text{ for all variables } x_i$$

Local extremum => critical point.

Example. Let f(x,y) = 3x^2 - 6xy + 2y^3. Find all the critical points of f (and whether they are local maxima, minima, or saddle points).

Solution. We solve $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

Notation: $f_x = \frac{\partial f}{\partial x}$ $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$

$\frac{\partial f}{\partial x} = 6x - 6y = 0$ $\frac{\partial f}{\partial y} = -6x + 6y^2$.

Eqn 1: $y = x$ Eqn 2: $x = y^2$

so $y = y^2 = y^2 - y = y(y-1) = 0$, $y = 0$ or $1$

$(x,y) = (0,0)$ or $(1,1)$.

These are the two critical points.

Find the Taylor polynomial of degree 2 of f(x,y) = 3x^2 - 6xy + 2y^3 about (x,y) = (0,0).

Solution: $\frac{\partial^2 f}{\partial x^2} = 6$ $\frac{\partial^2 f}{\partial y \partial x} = -6$ $\frac{\partial^2 f}{\partial y^2} = 12y$

at $(0,0)$ these are $6, -6, 0$.

Taylor polynomial of deg 2

$= f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2} f_{xx}(0,0)x^2$
$+ f_{xy}(0,0)xy + \frac{1}{2} f_{yy}(0,0)y^2$

$= 3x^2 - 6xy$.

The Hessian matrix is $\begin{bmatrix} 6 & -6 \\ -6 & 0 \end{bmatrix}$

Notice that 3x^2 - 6xy = 3[ (x-y)^2 - y^2 ] is a saddle point. On $y = 0$ we are positive. On $x = y$ we are negative.

Find the Taylor polynomial of degree 2 of
f(x,y) = 3x^2 - 6xy + 2y^3 about (x,y) = (1,1).

The Hessian matrix is $\begin{bmatrix} 6 & -6 \\ -6 & 12 \end{bmatrix}$

Put $u = x-1$ $v = y-1$

Notice that
3u^2 - 6uv + 6v^2 = quadratic part of deg 2 Taylor poly about $(x,y) = (1,1)$

= 3[ (u-v)^2 + v^2], which has a minimum
when (u,v) = (0,0).

## The second derivative test for local extrema.

If $a$ is a critical point of $f$ then it is either a local minimum, a local maximum, or a saddle point.

Sufficient criterion for a local minimum:

$f_{xx}(a) > 0$ and $\det H > 0$

Sufficient criterion for a local maximum:

$f_{xx}(a) < 0$ and $\det H > 0$

Sufficient criterion for a saddle point:

$\det H < 0$ . and $f_{xx} \neq 0$ .

In other cases, the test is inconclusive, we can't tell. This happens when

Let's try this out with our Hessian matrices:

$$H = \begin{array}{c} \quad f_{xx} \\ \begin{bmatrix} 0 & -6 \\ -6 & 0 \end{bmatrix} \end{array}$$

$\det H = -36 < 0$ saddle point

$$H = \begin{array}{c} \quad f_{xx} \\ \begin{bmatrix} 6 & -6 \\ -6 & 12 \end{bmatrix} \end{array}$$

$\det H = 72 - 36 = 36 > 0$

minimum.

# Pre-class Warm-up!!!

Do you remember the criterion on the Hessian matrix for a local minimum / local maximum / saddle point?

If the Hessian matrix is

$$H = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

do we get a

saddle point

$f_{xx} = 3 > 0$

$\det H = -1 < 0$

a. Local minimum

b. Local maximum

c. Saddle point

d. Can't tell

What about the matrices:

$$H = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \qquad H = \begin{bmatrix} -3 & 2 \\ 2 & -6 \end{bmatrix}$$

minimum

$$H = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

e.g. $3x^2 + 2y^2$

saddle point

$$H = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

e.g. $3x^2 - 2y^2$

$$H = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$$

maximum, $-3x^2 - 2y^2$

Example $f(x,y) = 2(x+2y)^2 - (x-y)^2$

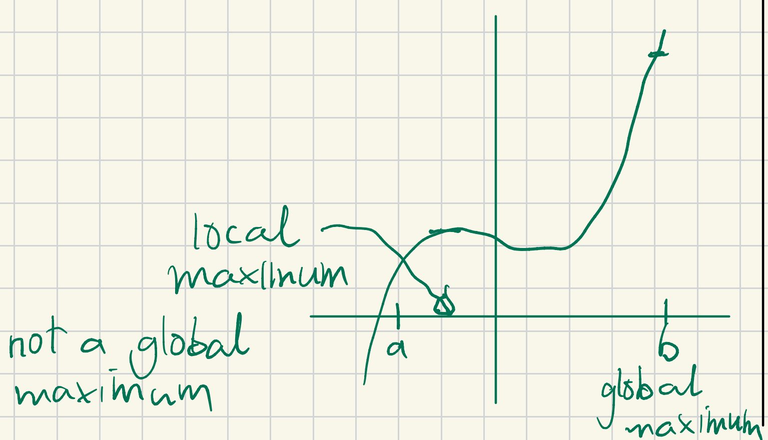
This has $f_{xx} > 0$ $f_{yy} > 0$ but is a saddle point.

We did a test for local extrema when there are two variables, and there is a more general test for more variables.

## Global maxima and minima

We learn

- on a closed bounded region of R^n a continuous function always has a point where it takes a maximum value and a point where it takes a minimum value. (There may be more than one such point.)
- What do bounded and closed mean? What is the boundary? What is the interior?
- How to find global extrema.

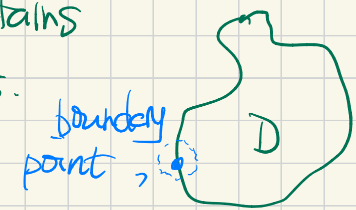


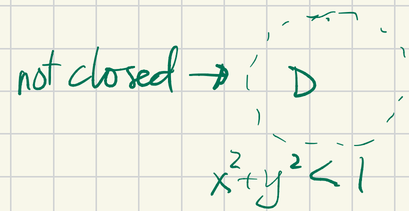

## Definitions of boundary, closed, open

There are different, equivalent definitions

$v$ is a boundary point of a region $D$ in $\mathbb{R}^n$ if every arbitrarily small ball in $\mathbb{R}^n$, center $v$, contains both a point of $D$ and a point not in $D$.

$D$ is "closed" if it contains all its boundary points.



To find global extrema:

1. Check for local extrema
2. Find the max and min values of f on the boundary of D
3. Compare the values we get at local extrema with those on the boundary

Our approach for 2: parametrize the boundary, then put derivatives $= 0$

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$\frac{d}{dt}(2-\cos t-\sin t) = \sin t - \cos t = 0$

$\sin t = \cos t$? $\quad t = \frac{\pi}{4}, \frac{5\pi}{4}$

The values of f at $(\frac{1}{2},\frac{1}{2}), (\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$

are $\underbrace{\frac{1}{2}}_{\text{smallest}}, \quad 2-\sqrt{2}, \quad \underbrace{2+\sqrt{2}}_{\text{biggest}}$

Example: Find the maximum and minimum values of
f(x,y) = x^2 + y^2 - x - y + 1
on the unit disk x^2 + y^2 ≤ 1.

Solution Step 1. Check local extrema when $x^2+y^2<1$

Step 2. find extrema on $x^2+y^2=1$

Step 3 compare

Step 1. $\frac{\partial f}{\partial x} = 2x-1 = 0 \quad x=\frac{1}{2}$

$\frac{\partial f}{\partial y} = 2y-1, \quad y=\frac{1}{2}$. Critical point $(\frac{1}{2},\frac{1}{2})$

$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. minimum.

Step 2. Parametrize the boundary $c(t)=(\cos(t),\sin(t))$

f takes the same values as $f(c(t))$ on the boundary.

find the max/min of $f(\cos t, \sin t)$

$= 1-\cos t-\sin t+1 = 2-\cos t-\sin t$.